Independence and abstract multiplication *

Dov M Gabbay †
King's College, London ‡
and
Bar-Ilan University, Israel §
and
University of Luxembourg ¶

Karl Schlechta ‖
Laboratoire d'Informatique Fondamentale de Marseille **

January 18, 2010

Abstract

We investigate the notion of independence, which is at the basis of many, seemingly unrelated, properties of logics, like the Rational Monotony rule of nonmonotonic logics, but also of interpolation theorems of monotonic and nonmonotonic logic. We show a strong connection between independence and certain rules about multiplication of abstract size in the field of nonmonotonic logic. We think that this notion of independence, with its ramifications, is extremely important, and has not been sufficiently investigated.

Contents

*paper 367

†Dov.Gabbay@kcl.ac.uk, www.dcs.kcl.ac.uk/staff/dg
‡Department of Computer Science, King's College London, Strand, London WC2R 2LS, UK
§Department of Computer Science, Bar-Ilan University, 52900 Ramat-Gan, Israel
¶Computer Science and Communications, Faculty of Sciences, 6, rue Coudenhove-Kalergi, L-1359 Luxembourg
‖ks@cmi.univ-mrs.fr, karl.schlechta@web.de, http://www.cmi.univ-mrs.fr/ ∼ ks
**UMR 6166, CNRS and Université de Provence, Address: CMI, 39, rue Joliot-Curie, F-13453 Marseille Cedex 13, France

1 Abstract definition of independence

Diagram 1.1

$$
\begin{tikzcd}
\Sigma_1 \arrow[rr] \arrow[dr] & & f(\Sigma_1) \arrow[dr, leftrightarrow] & \\
& \Sigma_1 \circ \Sigma_2 \arrow[rr] & & f(\Sigma_1 \circ \Sigma_2) = f(\Sigma_1) \circ' f(\Sigma_2) \\
\Sigma_2 \arrow[rr] \arrow[ur] & & f(\Sigma_2) \arrow[ur, leftrightarrow] &
\end{tikzcd}
$$

Note that $\circ$ and $\circ'$ might be different

Independence

The right notion of independence in our context seems to be:

We have compositions $\circ$ and $\circ'$, and operation f. We can calculate $f(\Sigma_1 \circ \Sigma_2)$ from $f(\Sigma_1)$ and $f(\Sigma_2)$, but also conversely, given $f(\Sigma_1 \circ \Sigma_2)$ we can calculate $f(\Sigma_1)$ and $f(\Sigma_2)$. Of course, in other contexts, other notions of independence might be adequate. More precisely:

Definition 1.1

Let $f : \mathcal{D} \to \mathcal{C}$ be any function from domain $\mathcal{D}$ to co-domain $\mathcal{C}$. Let $\circ$ be a "composition function" $\circ : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$, likewise for $\circ' : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

We say that $\langle f, \circ, \circ' \rangle$ are independent iff for any $\Sigma_i \in \mathcal{D}$

(1) $f(\Sigma_1 \circ \Sigma_2) = f(\Sigma_1) \circ' f(\Sigma_2)$,

(2) we can recover $f(\Sigma_i)$ from $f(\Sigma_1 \circ \Sigma_2)$, provided we know how $\Sigma_1 \circ \Sigma_2$ splits into the Σ_i, without using f again.

1.1 Discussion

(1) Ranked structures satisfy it:

  Let $\circ = \circ' = \cup$. Let f be the minimal model operator μ of preferential logic. Let $X, Y \subseteq X \cup Y$ have (at least) medium size (see below). Then $\mu(X \cup Y) = \mu(X) \cup \mu(Y)$, and $\mu(X) = \mu(X \cup Y) \cap X$, $\mu(Y) = \mu(X \cup Y) \cap Y$.

(2) Consistent classical formulas and their interpretation satisfy it:

  Let $\circ$ be conjunction in the composed language, $\circ'$ be model set intersection, $f(\phi) = M(\phi)$. Let ϕ, ψ be classical formulas, defined on disjoint language fragments $\mathcal{L}$, $\mathcal{L}'$ of some language $\mathcal{L}''$. Then $f(\phi \wedge \psi) = M(\phi) \cap M(\psi)$, and $M(\phi)$ is the projection of $M(\phi) \cap M(\psi)$ onto the (models of) language $\mathcal{L}$, likewise for $M(\psi)$. This is due to the way validity is defined, using only variables which occur in the formula.

  As a consequence, monotonic logic has semantical interpolation - see [GS09c], and below, Section 4.3.1 (page 16). The definition of being insensitive is justified by this modularity.

(3) It does not hold for inconsistent classical formulas: We cannot recover $M(a \wedge \neg a)$ and $M(b)$ from $M(a \wedge \neg a \wedge b)$, as we do not know where the inconsistency came from. The basic reason is trivial: One empty factor suffices to make the whole product empty, and we do not know which factor was the culprit. See Section 4.5 (page 20) for the discussion of a remedy.

(4) Preferential logic satisfies it under certain conditions:

  If $\mu(X \times Y) = \mu(X) \times \mu(Y)$ holds for model products and $\mid\sim$, then it holds by definition. An important consequence is that such a logic has interpolation of the form $\mid\sim \circ \mid\sim$, see Section 4.3 (page 16).

(5) Modular revision a la Parikh is based on a similar idea.

1.2 Independence and multiplication of abstract size

We are mainly interested in nonmonotonic logic. In this domain, independence is strongly connected to multiplication of abstract size, and much of the present paper treats this connection and its repercussions.

We have at least two scenarios for multiplication, one is decribed in Diagram 3.1 (page 11), the second in Diagram 4.1 (page 12). In the first scenario, we have nested sets, in the second, we have set products. In the first scenario, we consider subsets which behave as the big set does, in the second scenario we consider subspaces, and decompose the behaviour of the big space into behaviour of the subspaces. In both cases, this results naturally in multiplication of abstract sizes. When we look at the corresponding relation properties, they are quite different (rankedness vs. some kind of modularity). But this is perhaps to be expected, as the two scenarios are quite different.

We do not know whether there are still other, interesting, scenarios to consider in our framework.

2 Introduction to abstract size, additive rules

To put our work more into perspective, we first repeat in this section material from [GS08c]. This gives the main definitions and rules for non-monotonic logics, see Table 1 (page 5) and Table 2 (page 8), "Logical rules, definitions and connections". We then give the main additive rules for manipulation of abstract size from [GS09a], see Table 3 (page 9) and Table 4 (page 10), "Rules on size".

Explanation of Table 1 (page 5), "Logical rules, definitions and connections Part I'' and Table 2 (page 8), "Logical rules, definitions and connections Part II":

The tables are split in two, as they would not fit onto a page otherwise. The difference between the first two columns is that the first column treats the formula version of the rule, the second the more general theory (i.e., set of formulas) version.

The first column "Corr." is to be understood as follows:

Let a logic $\mid\!\sim$ satisfy (LLE) and (CCL), and define a function $f : \boldsymbol{D}_{\mathcal{L}} \to \boldsymbol{D}_{\mathcal{L}}$ by $f(M(T)) := M(\overline{\overline{T}})$. Then f is well defined, satisfies (μdp), and $\overline{\overline{T}} = Th(f(M(T)))$.

If $\mid\!\sim$ satisfies a rule in the left hand side, then - provided the additional properties noted in the middle for $\Rightarrow$ hold, too - f will satisfy the property in the right hand side.

Conversely, if $f : \mathcal{Y} \to \mathcal{P}(M_{\mathcal{L}})$ is a function, with $\boldsymbol{D}_{\mathcal{L}} \subseteq \mathcal{Y}$, and we define a logic $\mid\!\sim$ by $\overline{\overline{T}} := Th(f(M(T)))$, then $\mid\!\sim$ satisfies (LLE) and (CCL). If f satisfies (μdp), then $f(M(T)) = M(\overline{\overline{T}})$.

If f satisfies a property in the right hand side, then - provided the additional properties noted in the middle for $\Leftarrow$ hold, too - $\mid\!\sim$ will satisfy the property in the left hand side.

We use the following abbreviations for those supplementary conditions in the "Correspondence" columns: "$T = \phi$" means that, if one of the theories (the one named the same way in Definition 2 (page 4)) is equivalent to a formula, we do not need (μdp). $-(\mu dp)$ stands for "without (μdp)".

$A = B \parallel C$ will abbreviate $A = B$, or $A = C$, or $A = B \cup C$.

2.1 Notation

(1) $\mathcal{P}(X)$ is the power set of X, $\subseteq$ is the subset relation, $\subset$ the strict part of $\subseteq$, i.e. $A \subset B$ iff $A \subseteq B$ and $A \neq B$. The operators $\wedge$, $\neg$, $\vee$, $\rightarrow$ and $\vdash$ have their usual, classical interpretation.

(2) $\mathcal{I}(X) \subseteq \mathcal{P}(X)$ and $\mathcal{F}(X) \subseteq \mathcal{P}(X)$ are dual abstract notions of size, $\mathcal{I}(X)$ is the set of "small" subsets of X, $\mathcal{F}(X)$ the set of "big" subsets of X. They are dual in the sense that $A \in \mathcal{I}(X) \Leftrightarrow X - A \in \mathcal{F}(X)$. "$\mathcal{I}$" evokes "ideal", "$\mathcal{F}$" evokes "filter" though the full strength of both is reached only in $(< \omega * s)$. "s" evokes "small", and "$(x * s)$" stands for "x small sets together are still not everything".

(3) If $A \subseteq X$ is neither in $\mathcal{I}(X)$, nor in $\mathcal{F}(X)$, we say it has medium size, and we define $\mathcal{M}(X) := \mathcal{P}(X) - (\mathcal{I}(X) \cup \mathcal{F}(X))$. $\mathcal{M}^+(X) := \mathcal{P}(X) - \mathcal{I}(X)$ is the set of subsets which are not small.

(4) $\nabla x \phi$ is a generalized first order quantifier, it is read "almost all x have property ϕ". $\nabla x(\phi : \psi)$ is the relativized version, read: "almost all x with property ϕ have also property ψ". To keep the table "Rules on size" simple, we write mostly only the non-relativized versions. Formally, we have $\nabla x \phi :\Leftrightarrow \{x : \phi(x)\} \in \mathcal{F}(U)$ where U is the universe, and $\nabla x(\phi : \psi) :\Leftrightarrow \{x : (\phi \wedge \psi)(x)\} \in \mathcal{F}(\{x : \phi(x)\})$. Soundness and completeness results on ∇ can be found in [Sch95-1].

(5) Analogously, for propositional logic, we define:

$\alpha \mid\!\sim \beta :\Leftrightarrow M(\alpha \wedge \beta) \in \mathcal{F}(M(\alpha))$,

where $M(\phi)$ is the set of models of ϕ.

Table 1: Logical rules, definitions and connections Part I

Logical rules, definitions and connections Part I					
Logical rule		Corr.	Model set	Corr.	Size Rules
Basics					
(SC) Supraclassicality $\alpha \vdash \beta \Rightarrow \alpha \mid\sim \beta$ (REF) Reflexivity $T \cup \{\alpha\} \mid\sim \alpha$	(SC) $\overline{T} \subseteq \overline{\overline{T}}$	$\Rightarrow$ $\Leftarrow$	$(\mu \subseteq)$ $f(X) \subseteq X$	trivial	(Opt)
(LLE) Left Logical Equivalence $\vdash \alpha \leftrightarrow \alpha', \alpha \mid\sim \beta \Rightarrow$ $\alpha' \mid\sim \beta$	(LLE) $\overline{T} = \overline{T'} \Rightarrow \overline{\overline{T}} = \overline{\overline{T'}}$				
(RW) Right Weakening $\alpha \mid\sim \beta, \vdash \beta \rightarrow \beta' \Rightarrow$ $\alpha \mid\sim \beta'$	(RW) $T \mid\sim \beta, \vdash \beta \rightarrow \beta' \Rightarrow$ $T \mid\sim \beta'$			trivial	(iM)
(wOR) $\alpha \mid\sim \beta,\ \alpha' \vdash \beta \Rightarrow$ $\alpha \vee \alpha' \mid\sim \beta$	(wOR) $\overline{\overline{T}} \cap \overline{T'} \subseteq \overline{\overline{T \vee T'}}$	$\Rightarrow$ $\Leftarrow$	(μwOR) $f(X \cup Y) \subseteq f(X) \cup Y$	$\Leftrightarrow$	$(eM\mathcal{I})$
$(disjOR)$ $\alpha \vdash \neg \alpha',\ \alpha \mid\sim \beta,$ $\alpha' \mid\sim \beta \Rightarrow \alpha \vee \alpha' \mid\sim \beta$	$(disjOR)$ $\neg Con(T \cup T') \Rightarrow$ $\overline{\overline{T}} \cap \overline{\overline{T'}} \subseteq \overline{\overline{T \vee T'}}$	$\Rightarrow$ $\Leftarrow$	$(\mu disjOR)$ $X \cap Y = \emptyset \Rightarrow$ $f(X \cup Y) \subseteq f(X) \cup f(Y)$	$\Leftrightarrow$	$(I \cup disj)$
(CP) Consistency Preservation $\alpha \mid\sim \bot \Rightarrow \alpha \vdash \bot$	(CP) $T \mid\sim \bot \Rightarrow T \vdash \bot$	$\Rightarrow$ $\Leftarrow$	$(\mu\emptyset)$ $f(X) = \emptyset \Rightarrow X = \emptyset$	trivial	(I_1)
			$(\mu\emptyset fin)$ $X \neq \emptyset \Rightarrow f(X) \neq \emptyset$ for finite X		(I_1)
	(AND_1) $\alpha \mid\sim \beta \Rightarrow \alpha \not\mid\sim \neg\beta$				(I_2)
	(AND_n) $\alpha \mid\sim \beta_1, \ldots, \alpha \mid\sim \beta_{n-1} \Rightarrow$ $\alpha \not\mid\sim (\neg\beta_1 \vee \ldots \vee \neg\beta_{n-1})$				(I_n)
(AND) $\alpha \mid\sim \beta, \alpha \mid\sim \beta' \Rightarrow$ $\alpha \mid\sim \beta \wedge \beta'$	(AND) $T \mid\sim \beta, T \mid\sim \beta' \Rightarrow$ $T \mid\sim \beta \wedge \beta'$			trivial	(I_ω)
(CCL) Classical Closure	(CCL) $\overline{\overline{T}}$ classically closed			trivial	$(iM) + (I_\omega)$
(OR) $\alpha \mid\sim \beta, \alpha' \mid\sim \beta \Rightarrow$ $\alpha \vee \alpha' \mid\sim \beta$	(OR) $\overline{\overline{T}} \cap \overline{\overline{T'}} \subseteq \overline{\overline{T \vee T'}}$	$\Rightarrow$ $\Leftarrow$	(μOR) $f(X \cup Y) \subseteq f(X) \cup f(Y)$	$\Leftrightarrow$	$(eM\mathcal{I}) + (I_\omega)$
$\overline{\overline{\alpha \wedge \alpha'}} \subseteq \overline{\overline{\overline{\alpha}} \cup \{\alpha'\}}$	(PR) $\overline{\overline{T \cup T'}} \subseteq \overline{\overline{\overline{T}} \cup T'}$	$\Rightarrow$ $\Leftarrow (\mu dp) + (\mu \subseteq)$ $\not\Leftarrow -(\mu dp)$ $\Leftarrow (\mu \subseteq)$ $T' = \phi$	(μPR) $X \subseteq Y \Rightarrow$ $f(Y) \cap X \subseteq f(X)$	$\Leftrightarrow$	$(eM\mathcal{I}) + (I_\omega)$
		$\Leftarrow$ $T' = \phi$	$(\mu PR')$ $f(X) \cap Y \subseteq f(X \cap Y)$		
(CUT) $T \mid\sim \alpha; T \cup \{\alpha\} \mid\sim \beta \Rightarrow$ $T \mid\sim \beta$	(CUT) $T \subseteq \overline{T'} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T'}} \subseteq \overline{\overline{T}}$	$\Rightarrow$ $\Leftarrow$	(μCUT) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(X) \subseteq f(Y)$	$\Leftarrow$ $\not\Rightarrow$	$(eM\mathcal{I}) + (I_\omega)$

(6) In preferential structures, $\mu(X) \subseteq X$ is the set of minimal elements of X. This generates a principal filter by $\mathcal{F}(X) := \{A \subseteq X : \mu(X) \subseteq A\}$. Corresponding properties about μ are not listed systematically.

(7) The usual rules (AND) etc. are named here (AND_ω), as they are in a natural ascending line of similar rules, based on strengthening of the filter/ideal properties.

(8) For any set of formulas T, and any consequence relation $\mid\sim$, we will use $\overline{T} := \{\phi : T \vdash \phi\}$, the set of classical consequences of T, and $\overline{\overline{T}} := \{\phi : T \mid\sim \phi\}$, the set of consequences of T under the relation $\mid\sim$.

(9) We say that a set X of models is definable by a formula (or a theory) iff there is a formula ϕ (a theory T) such that $X = M(\phi)$, or $X = M(T)$, the set of models of ϕ or T, respectively.

(10) Most rules are explained in the table "Logical rules", and "RW" stands for Right Weakening.

2.2 The groupes of rules

The rules concern properties of $\mathcal{I}(X)$ or $\mathcal{F}(X)$, or dependencies between such properties for different X and Y. All X, Y, etc. will be subsets of some universe, say V. Intuitively, V is the set of all models of some fixed propositional language. It is not necessary to consider all subsets of V, the intention is to consider subsets of V, which are definable by a formula or a theory. So we assume all X, Y etc. taken from some $\mathcal{Y} \subseteq \mathcal{P}(V)$, which we

call the domain. In the former case, $\mathcal{Y}$ is closed under set difference, in the latter case not necessarily so. (We will mention it when we need some particular closure property.)

The rules are divided into 5 groups:

(1) (Opt), which says that "All" is optimal - i.e. when there are no exceptions, then a soft rule $\mid\!\sim$ holds.

(2) 3 monotony rules:

  (2.1) (iM) is inner monotony, a subset of a small set is small,

  (2.2) $(eM\mathcal{I})$ external monotony for ideals: enlarging the base set keeps small sets small,

  (2.3) $(eM\mathcal{F})$ external monotony for filters: a big subset stays big when the base set shrinks.

  These three rules are very natural if "size" is anything coherent over change of base sets. In particular, they can be seen as weakening.

(3) $(\approx)$ keeps proportions, it is here mainly to point the possibility out.

(4) a group of rules $x * s$, which say how many small sets will not yet add to the base set. The notation "$(< \omega * s)$" is an allusion to the full filter property, that filters are closed under $finite$ intersections.

(5) Rational monotony, which can best be understood as robustness of $\mathcal{M}^+$, see $(\mathcal{M}^{++})(3)$.

We will assume all base sets to be non-empty in order to avoid pathologies and in particular clashes between (Opt) and $(1 * s)$.

Note that the full strength of the usual definitions of a filter and an ideal are reached only in line $(< \omega * s)$.

2.2.1 Regularities

(1) The group of rules $(x * s)$ use ascending strength of $\mathcal{I}/\mathcal{F}$.

(2) The column $(\mathcal{M}^+)$ contains interesting algebraic properties. In particular, they show a strengthening from $(3 * s)$ up to Rationality. They are not necessarily equivalent to the corresponding (I_x) rules, not even in the presence of the basic rules. The examples show that care has to be taken when considering the different variants.

(3) Adding the somewhat superflous (CM_2), we have increasing cautious monotony from (wCM) to full (CM_ω).

(4) We have increasing "or" from (wOR) to full (OR_ω).

(5) The line $(2 * s)$ is only there because there seems to be no $(\mathcal{M}_2^+)$, otherwise we could begin $(n * s)$ at $n = 2$.

2.2.2 Summary

We can obtain all rules except $(RatM)$ and $(\approx)$ from (Opt), the monotony rules - (iM), $(eM\mathcal{I})$, $(eM\mathcal{F})$ -, and $(x * s)$ with increasing x.

2.3 Table

The following table is split in two, as it is too big for printing in one page.

(See Table 3 (page 9), "Rules on size - Part I'' and Table 4 (page 10), "Rules on size - Part II".

Logical rules, definitions and connections Part II					
Logical rule		Corr.	Model set	Corr.	Size-Rule
Cumulativity					
(wCM) $\alpha \mid\!\sim \beta, \alpha' \vdash \alpha, \alpha \wedge \beta \vdash \alpha' \Rightarrow$ $\alpha' \mid\!\sim \beta$				trivial	$(eM\mathcal{F})$
(CM_2) $\alpha \mid\!\sim \beta, \alpha \mid\!\sim \beta' \Rightarrow \alpha \wedge \beta \not\vdash \neg\beta'$					(I_2)
(CM_n) $\alpha \mid\!\sim \beta_1, \ldots, \alpha \mid\!\sim \beta_n \Rightarrow$ $\alpha \wedge \beta_1 \wedge \ldots \wedge \beta_{n-1} \not\vdash \neg\beta_n$					(I_n)
(CM) Cautious Monotony $\alpha \mid\!\sim \beta, \alpha \mid\!\sim \beta' \Rightarrow$ $\alpha \wedge \beta \mid\!\sim \beta'$	(CM) $T \subseteq \overline{T'} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T}} \subseteq \overline{\overline{T'}}$	$\Rightarrow$ $\Leftarrow$	(μCM) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(Y) \subseteq f(X)$	$\Leftrightarrow$	$(\mathcal{M}^+_\omega)(4)$
or $(ResM)$ Restricted Monotony $T \mid\!\sim \alpha, \beta \Rightarrow T \cup \{\alpha\} \mid\!\sim \beta$		$\Rightarrow$ $\Leftarrow$	$(\mu ResM)$ $f(X) \subseteq A \cap B \Rightarrow$ $f(X \cap A) \subseteq B$		
(CUM) Cumulativity $\alpha \mid\!\sim \beta \Rightarrow$ $(\alpha \mid\!\sim \beta' \Leftrightarrow \alpha \wedge \beta \mid\!\sim \beta')$	(CUM) $T \subseteq \overline{T'} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T}} = \overline{\overline{T'}}$	$\Rightarrow$ $\Leftarrow$	(μCUM) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(Y) = f(X)$	$\Leftarrow$ $\not\Rightarrow$	$(eM\mathcal{I}) + (I_\omega) + (\mathcal{M}^+_\omega)(4)$
	$(\subseteq\supseteq)$ $T \subseteq \overline{\overline{T'}}, T' \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T'}} = \overline{\overline{T}}$	$\Rightarrow$ $\Leftarrow$	$(\mu \subseteq\supseteq)$ $f(X) \subseteq Y, f(Y) \subseteq X \Rightarrow$ $f(X) = f(Y)$	$\Leftarrow$ $\not\Rightarrow$	$(eM\mathcal{I}) + (I_\omega) + (eM\mathcal{F})$
Rationality					
$(RatM)$ Rational Monotony $\alpha \mid\!\sim \beta, \alpha \not\mid\!\sim \neg\beta' \Rightarrow$ $\alpha \wedge \beta' \mid\!\sim \beta$	$(RatM)$ $Con(T \cup \overline{\overline{T'}}), T \vdash T' \Rightarrow$ $\overline{\overline{T}} \supseteq \overline{\overline{T'}} \cup T$	$\Rightarrow$ $\Leftarrow (\mu dp)$ $\not\Leftarrow -(\mu dp)$ $\Leftarrow T = \phi$	$(\mu RatM)$ $X \subseteq Y, X \cap f(Y) \neq \emptyset \Rightarrow$ $f(X) \subseteq f(Y) \cap X$	$\Leftrightarrow$	$(\mathcal{M}^{++})$
	$(RatM =)$ $Con(T \cup \overline{\overline{T'}}), T \vdash T' \Rightarrow$ $\overline{\overline{T}} = \overline{\overline{T'}} \cup T$	$\Rightarrow$ $\Leftarrow (\mu dp)$ $\not\Leftarrow -(\mu dp)$ $\Leftarrow T = \phi$	$(\mu =)$ $X \subseteq Y, X \cap f(Y) \neq \emptyset \Rightarrow$ $f(X) = f(Y) \cap X$		
	$(Log =')$ $Con(\overline{\overline{T'}} \cup T) \Rightarrow$ $\overline{\overline{T \cup T'}} = \overline{\overline{T'}} \cup T$	$\Rightarrow$ $\Leftarrow (\mu dp)$ $\not\Leftarrow -(\mu dp)$ $\Leftarrow T = \phi$	$(\mu =')$ $f(Y) \cap X \neq \emptyset \Rightarrow$ $f(Y \cap X) = f(Y) \cap X$		
(DR) $\alpha \vee \beta \mid\!\sim \gamma \Rightarrow$ $\alpha \mid\!\sim \gamma$ or $\beta \mid\!\sim \gamma$	$(Log \Vert)$ $\overline{\overline{T \vee T'}}$ is one of $\overline{\overline{T}}$, or $\overline{\overline{T'}}$, or $\overline{\overline{T}} \cap \overline{\overline{T'}}$ (by (CCL))	$\Rightarrow$ $\Leftarrow$	$(\mu \Vert)$ $f(X \cup Y)$ is one of $f(X)$, $f(Y)$ or $f(X) \cup f(Y)$		
	$(Log\cup)$ $Con(\overline{\overline{T'}} \cup T), \neg Con(\overline{\overline{T'}} \cup \overline{\overline{T}}) \Rightarrow$ $\neg Con(\overline{\overline{T \vee T'}} \cup T')$	$\Rightarrow (\mu \subseteq) + (\mu =)$ $\Leftarrow (\mu dp)$ $\not\Leftarrow -(\mu dp)$	$(\mu\cup)$ $f(Y) \cap (X - f(X)) \neq \emptyset \Rightarrow$ $f(X \cup Y) \cap Y = \emptyset$		
	$(Log\cup')$ $Con(\overline{\overline{T'}} \cup T), \neg Con(\overline{\overline{T'}} \cup \overline{\overline{T}}) \Rightarrow$ $\overline{\overline{T \vee T'}} = \overline{\overline{T}}$	$\Rightarrow (\mu \subseteq) + (\mu =)$ $\Leftarrow (\mu dp)$ $\not\Leftarrow -(\mu dp)$	$(\mu\cup')$ $f(Y) \cap (X - f(X)) \neq \emptyset \Rightarrow$ $f(X \cup Y) = f(X)$		
			$(\mu \in)$ $a \in X - f(X) \Rightarrow$ $\exists b \in X. a \not\in f(\{a, b\})$		

Table 2: Logical rules, definitions and connections Part II

Table 3: Rules on size - Part I

Rules on size - Part I				
	"Ideal"	"Filter"	$\mathcal{M}^+$	∇
Optimal proportion				
(Opt)	$\emptyset \in \mathcal{I}(X)$	$X \in \mathcal{F}(X)$		$\forall x\alpha \to \nabla x\alpha$
Monotony (Improving proportions). (iM): internal monotony, $(eM\mathcal{I})$: external monotony for ideals, $(eM\mathcal{F})$: external monotony for filters				
(iM)	$A \subseteq B \in \mathcal{I}(X)$ $\Rightarrow$ $A \in \mathcal{I}(X)$	$A \in \mathcal{F}(X)$, $A \subseteq B \subseteq X$ $\Rightarrow B \in \mathcal{F}(X)$		$\nabla x\alpha \wedge \forall x(\alpha \to \alpha')$ $\to \nabla x\alpha'$
$(eM\mathcal{I})$	$X \subseteq Y \Rightarrow$ $\mathcal{I}(X) \subseteq \mathcal{I}(Y)$			$\nabla x(\alpha : \beta) \wedge$ $\forall x(\alpha' \to \beta) \to$ $\nabla x(\alpha \vee \alpha' : \beta)$
$(eM\mathcal{F})$		$X \subseteq Y \Rightarrow$ $\mathcal{F}(Y) \cap \mathcal{P}(X) \subseteq$ $\mathcal{F}(X)$		$\nabla x(\alpha : \beta) \wedge$ $\forall x(\beta \wedge \alpha \to \alpha') \to$ $\nabla x(\alpha \wedge \alpha' : \beta)$
Keeping proportions				
$(\approx)$	$(\mathcal{I} \cup disj)$ $A \in \mathcal{I}(X)$, $B \in \mathcal{I}(Y)$, $X \cap Y = \emptyset \Rightarrow$ $A \cup B \in \mathcal{I}(X \cup Y)$	$(\mathcal{F} \cup disj)$ $A \in \mathcal{F}(X)$, $B \in \mathcal{F}(Y)$, $X \cap Y = \emptyset \Rightarrow$ $A \cup B \in \mathcal{F}(X \cup Y)$	$(\mathcal{M}^+ \cup disj)$ $A \in \mathcal{M}^+(X)$, $B \in \mathcal{M}^+(Y)$, $X \cap Y = \emptyset \Rightarrow$ $A \cup B \in \mathcal{M}^+(X \cup Y)$	$\nabla x(\alpha : \beta) \wedge$ $\nabla x(\alpha' : \beta) \wedge$ $\neg\exists x(\alpha \wedge \alpha') \to$ $\nabla x(\alpha \vee \alpha' : \beta)$
Robustness of proportions: $n * small \neq All$				
$(1 * s)$	$(\mathcal{I}_1)$ $X \notin \mathcal{I}(X)$	$(\mathcal{F}_1)$ $\emptyset \notin \mathcal{F}(X)$		(∇_1) $\nabla x\alpha \to \exists x\alpha$
$(2 * s)$	$(\mathcal{I}_2)$ $A, B \in \mathcal{I}(X) \Rightarrow$ $A \cup B \neq X$	$(\mathcal{F}_2)$ $A, B \in \mathcal{F}(X) \Rightarrow$ $A \cap B \neq \emptyset$		(∇_2) $\nabla x\alpha \wedge \nabla x\beta$ $\to \exists x(\alpha \wedge \beta)$
$(n * s)$ $(n \geq 3)$	$(\mathcal{I}_n)$ $A_1, ., A_n \in \mathcal{I}(X)$ $\Rightarrow$ $A_1 \cup . \cup A_n \neq X$	$(\mathcal{F}_n)$ $A_1, ., A_n \in \mathcal{I}(X)$ $\Rightarrow$ $A_1 \cap . \cap A_n \neq \emptyset$	$(\mathcal{M}^+_n)$ $X_1 \in \mathcal{F}(X_2), .,$ $X_{n-1} \in \mathcal{F}(X_n) \Rightarrow$ $X_1 \in \mathcal{M}^+(X_n)$	(∇_n) $\nabla x\alpha_1 \wedge . \wedge \nabla x\alpha_n$ $\to$ $\exists x(\alpha_1 \wedge . \wedge \alpha_n)$
$(< \omega * s)$	$(\mathcal{I}_\omega)$ $A, B \in \mathcal{I}(X) \Rightarrow$ $A \cup B \in \mathcal{I}(X)$	$(\mathcal{F}_\omega)$ $A, B \in \mathcal{F}(X) \Rightarrow$ $A \cap B \in \mathcal{F}(X)$	$(\mathcal{M}^+_\omega)$ (1) $A \in \mathcal{F}(X),\ X \in \mathcal{M}^+(Y)$ $\Rightarrow A \in \mathcal{M}^+(Y)$ (2) $A \in \mathcal{M}^+(X),\ X \in \mathcal{F}(Y)$ $\Rightarrow A \in \mathcal{M}^+(Y)$ (3) $A \in \mathcal{F}(X),\ X \in \mathcal{F}(Y)$ $\Rightarrow A \in \mathcal{F}(Y)$ (4) $A, B \in \mathcal{I}(X) \Rightarrow$ $A - B \in \mathcal{I}(X{-}B)$	(∇_ω) $\nabla x\alpha \wedge \nabla x\beta \to$ $\nabla x(\alpha \wedge \beta)$
Robustness of $\mathcal{M}^+$				
$(\mathcal{M}^{++})$			$(\mathcal{M}^{++})$ (1) $A \in \mathcal{I}(X),\ B \notin \mathcal{F}(X)$ $\Rightarrow A - B \in \mathcal{I}(X - B)$ (2) $A \in \mathcal{F}(X), B \notin \mathcal{F}(X)$ $\Rightarrow A - B \in \mathcal{F}(X - B)$ (3) $A \in \mathcal{M}^+(X)$, $X \in \mathcal{M}^+(Y)$ $\Rightarrow A \in \mathcal{M}^+(Y)$	

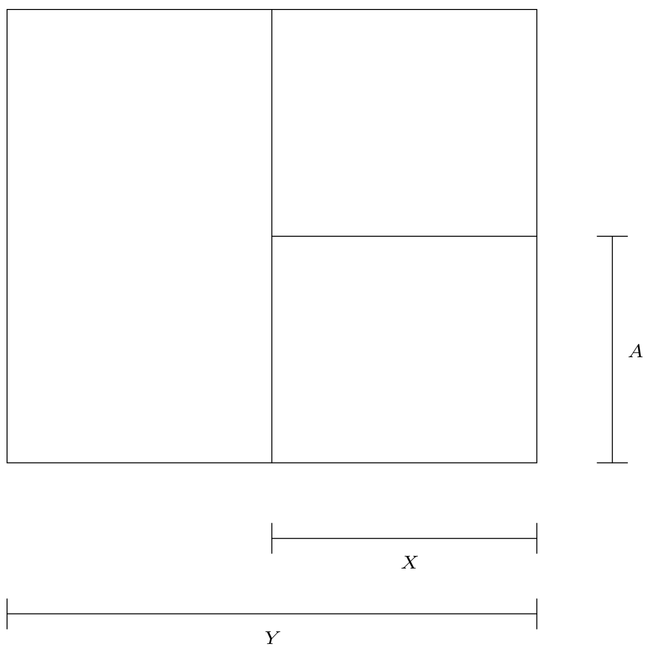
Table 4: Rules on size - Part II

Rules on size - Part II				
	various rules	AND	OR	Caut./Rat.Mon.
Optimal proportion				
(Opt)	(SC) $\alpha \vdash \beta \Rightarrow \alpha \mid\!\sim \beta$			
Monotony (Improving proportions)				
(iM)	(RW) $\alpha \mid\!\sim \beta, \beta \vdash \beta' \Rightarrow$ $\alpha \mid\!\sim \beta'$			
$(eM\mathcal{I})$	(PR') $\alpha \mid\!\sim \beta, \alpha \vdash \alpha',$ $\alpha' \wedge \neg\alpha \vdash \beta \Rightarrow$ $\alpha' \mid\!\sim \beta$ (μPR) $X \subseteq Y \Rightarrow$ $\mu(Y) \cap X \subseteq \mu(X)$		(wOR) $\alpha \mid\!\sim \beta,\ \alpha' \vdash \beta \Rightarrow$ $\alpha \vee \alpha' \mid\!\sim \beta$ (μwOR) $\mu(X \cup Y) \subseteq \mu(X) \cup Y$	
$(eM\mathcal{F})$				(wCM) $\alpha \mid\!\sim \beta, \alpha' \vdash \alpha,$ $\alpha \wedge \beta \vdash \alpha' \Rightarrow$ $\alpha' \mid\!\sim \beta$
Keeping proportions				
$(\approx)$	(NR) $\alpha \mid\!\sim \beta \Rightarrow$ $\alpha \wedge \gamma \mid\!\sim \beta$ or $\alpha \wedge \neg\gamma \mid\!\sim \beta$		$(disjOR)$ $\alpha \mid\!\sim \beta,\ \alpha' \mid\!\sim \beta'$ $\alpha \vdash \neg\alpha',\ \Rightarrow$ $\alpha \vee \alpha' \mid\!\sim \beta \vee \beta'$ $(\mu disjOR)$ $X \cap Y = \emptyset \Rightarrow$ $\mu(X \cup Y) \subseteq \mu(X) \cup \mu(Y)$	
Robustness of proportions: $n * small \neq All$				
$(1 * s)$	(CP) $\alpha \mid\!\sim \bot \Rightarrow \alpha \vdash \bot$	(AND_1) $\alpha \mid\!\sim \beta \Rightarrow \alpha \nvdash \neg\beta$		
$(2 * s)$		(AND_2) $\alpha \mid\!\sim \beta,\ \alpha \mid\!\sim \beta' \Rightarrow$ $\alpha \nvdash \neg\beta \vee \neg\beta'$	(OR_2) $\alpha \mid\!\sim \beta \Rightarrow \alpha \not\mid\!\sim \neg\beta$	(CM_2) $\alpha \mid\!\sim \beta \Rightarrow \alpha \not\mid\!\sim \neg\beta$
$(n * s)$ $(n \geq 3)$		(AND_n) $\alpha \mid\!\sim \beta_1, .., \alpha \mid\!\sim \beta_n$ $\Rightarrow$ $\alpha \nvdash \neg\beta_1 \vee . \vee \neg\beta_n$	(OR_n) $\alpha_1 \mid\!\sim \beta, .., \alpha_{n-1} \mid\!\sim \beta$ $\Rightarrow$ $\alpha_1 \vee . \vee \alpha_{n-1} \not\mid\!\sim \neg\beta$	(CM_n) $\alpha \mid\!\sim \beta_1, .., \alpha \mid\!\sim \beta_{n-1}$ $\Rightarrow$ $\alpha \wedge \beta_1 \wedge . \wedge \beta_{n-2} \not\mid\!\sim$ $\neg\beta_{n-1}$
$(< \omega * s)$		(AND_ω) $\alpha \mid\!\sim \beta,\ \alpha \mid\!\sim \beta' \Rightarrow$ $\alpha \mid\!\sim \beta \wedge \beta'$	(OR_ω) $\alpha \mid\!\sim \beta,\ \alpha' \mid\!\sim \beta \Rightarrow$ $\alpha \vee \alpha' \mid\!\sim \beta$ (μOR) $\mu(X \cup Y) \subseteq \mu(X) \cup \mu(Y)$	(CM_ω) $\alpha \mid\!\sim \beta,\ \alpha \mid\!\sim \beta' \Rightarrow$ $\alpha \wedge \beta \mid\!\sim \beta'$ (μCM) $\mu(X) \subseteq Y \subseteq X \Rightarrow$ $\mu(Y) \subseteq \mu(X)$
Robustness of $\mathcal{M}^+$				
$(\mathcal{M}^{++})$				$(RatM)$ $\alpha \mid\!\sim \beta, \alpha \not\mid\!\sim \neg\beta' \Rightarrow$ $\alpha \wedge \beta' \mid\!\sim \beta$ $(\mu RatM)$ $X \subseteq Y,$ $X \cap \mu(Y) \neq \emptyset \Rightarrow$ $\mu(X) \subseteq \mu(Y) \cap X$

3 Multiplication of size for subsets

Here we have nested sets, $A \subseteq X \subseteq Y$, A is a certain proportion of X, and X of Y, resulting in a multiplication of relative size or proportions. This is a classical subject of nonmonotonic logic, see the last section, taken from [GS09a], it is partly repeated here to stress the common points with the other scenario.

Diagram 3.1



Scenario 1

3.1 Properties

Diagram 3.1 (page 11) is to be read as follows: The whole set Y is split in X and $Y-X$, X is split in A and $X-A$. X is a small/medium/big part of Y, A is a small/medium/big part of X. The question is: is A a small/medium/big part of Y?

Note that the relation of A to X is conceptually different from that of X to Y, as we change the base set by going from X to Y, but not when going from A to X. Thus, in particular, when we read the diagram as expressing multiplication, commutativity is not necessarily true.

We looked at this scenario already in [GS09a], but there from an additive point of view, using various basic properties like (iM), $(eM\mathcal{I})$, $(eM\mathcal{F})$. Here, we use just multiplication - except sometimes for motivation.

We examine different rules:

If $Y = X$ or $X = A$, there is nothing to show, so 1 is the neutral element of multiplication.

If $X \in \mathcal{I}(Y)$ or $A \in \mathcal{I}(X)$, then we should have $A \in \mathcal{I}(Y)$. (Use for motivation (iM) or $(eM\mathcal{I})$ respectively.)

So it remains to look at the following cases, with the "natural" answers given already:

(1) $X \in \mathcal{F}(Y)$, $A \in \mathcal{F}(X) \Rightarrow A \in \mathcal{F}(Y)$,

(2) $X \in \mathcal{M}^+(Y)$, $A \in \mathcal{F}(X) \Rightarrow A \in \mathcal{M}^+(Y)$,

(3) $X \in \mathcal{F}(Y)$, $A \in \mathcal{M}^+(X) \Rightarrow A \in \mathcal{M}^+(Y)$,

(4) $X \in \mathcal{M}^+(Y)$, $A \in \mathcal{M}^+(X) \Rightarrow A \in \mathcal{M}^+(Y)$.

But (1) is case (3) of $(\mathcal{M}^+_\omega)$ in [GS09a], see Table "Rules on size" in Section 2 (page 4).

(2) is case (2) of $(\mathcal{M}^+_\omega)$ there,

(3) is case (1) of $(\mathcal{M}^+_\omega)$ there, finally,

(4) is $(\mathcal{M}^{++})$ there.

So the first three correspond to various expressions of (AND_ω), (OR_ω), (CM_ω), the last one to $(RatM)$.

But we can read them also the other way round, e.g.:

(1) corresponds to: $\alpha \mid\sim \beta$, $\alpha \wedge \beta \mid\sim \gamma \Rightarrow \alpha \mid\sim \gamma$,

(2) corresponds to: $\alpha \not\mid\sim \neg\beta$, $\alpha \wedge \beta \mid\sim \gamma \Rightarrow \alpha \not\mid\sim \neg(\beta \wedge \gamma)$,

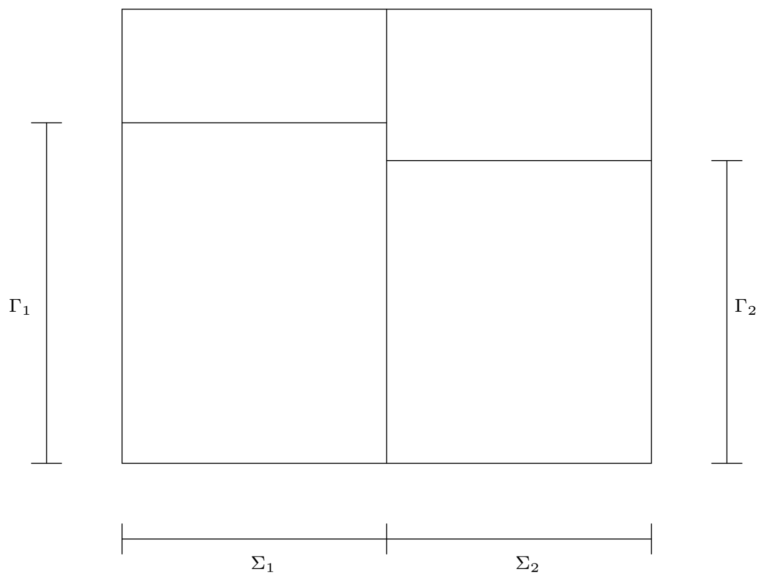
(3) corresponds to: $\alpha \mid\sim \beta$, $\alpha \wedge \beta \not\mid\sim \neg\gamma \Rightarrow \alpha \not\mid\sim \neg(\beta \wedge \gamma)$.

All these rules might be seen as too idealistic, so just as we did in [GS09a], we can consider milder versions: We might for instance consider a rule which says that $big * \ldots * big$, n times, is not small. Consider for instance the case $n = 2$. So we would conclude that A is not small in Y. In terms of logic, we then have: $\alpha \mid\sim \beta$, $\alpha \wedge \beta \mid\sim \gamma$ $\Rightarrow \alpha \not\mid\sim (\neg\beta \vee \neg\gamma)$. We can obtain the same logical property from $3 * small \neq all$.

4 Multiplication of size for subspaces

4.1 Properties

Diagram 4.1



Scenario 2

In this scenario, Σ_i are sets of sequences, see Diagram 4.1 (page 12). (Correponding, intuitively, to a set of models in language $\mathcal{L}_i$, Σ_i will be the set of α_i-models, and the subsets Γ_i are to be seen as the "best" models, where β_i will hold. The languages are supposed to be disjoint sublanguages of a common language $\mathcal{L}$.)

In this scenario, the Σ_i have symmetrical roles, so there is no intuitive reason for multiplication not to be commutative.

We can interpret the situation twofold:

First, we work separately in sublanguage $\mathcal{L}_1$ and $\mathcal{L}_2$, and, say, α_i and β_i are both defined in $\mathcal{L}_i$, and we look at $\alpha_i \mid\!\sim \beta_i$ in the sublanguage $\mathcal{L}_i$, or, we consider both α_i and β_i in the big language $\mathcal{L}$, and look at $\alpha_i \mid\!\sim \beta_i$ in $\mathcal{L}$. These two ways are a priori completely different. Speaking in preferential terms, it is not at all clear why the orderings on the submodels should have anything to do with the orderings on the whole models. It seems a very desirable property, but we have to postulate it, which we do now (an overview is given in Table 6 (page 23)). We give now informally a list of such rules, mainly to show the connection with the first scenario. Later, see Definition 4.1 (page 14), we will introduce formally some rules for which we show a connection with interpolation. Here, e.g., "$(big * big \Rightarrow big)$" stands for "if both factors are big, so will be the product", this will be abbreviated by "$b * b \Rightarrow b$" in Table 6 (page 23) .

$(big * 1 \Rightarrow big)$ Let $\Gamma_1 \subseteq \Sigma_1$, if $\Gamma_1 \in \mathcal{F}(\Sigma_1)$, then $\Gamma_1 \times \Sigma_2 \in \mathcal{F}(\Sigma_1 \times \Sigma_2)$, (and the dual rule for Σ_2 and Γ_2).

This property preserves proportions, so it seems intuitively quite uncontested, whenever we admit coherence over products. (Recall that there was nothing to show in the first scenario.)

When we re-consider above case: suppose $\alpha \mid\!\sim \beta$ in the sublanguage, so $M(\beta) \in \mathcal{F}(M(\alpha))$ in the sublanguage, so by $(big * 1 \Rightarrow big)$, $M(\beta) \in \mathcal{F}(M(\alpha))$ in the big language $\mathcal{L}$.

We obtain the dual rule for small (and likewise, medium size) sets:

$(small * 1 \Rightarrow small)$ Let $\Gamma_1 \subseteq \Sigma_1$, if $\Gamma_1 \in \mathcal{I}(\Sigma_1)$, then $\Gamma_1 \times \Sigma_2 \in \mathcal{I}(\Sigma_1 \times \Sigma_2)$, (and the dual rule for Σ_2 and Γ_2),

establishing $All = 1$ as the neutral element for multiplication.

We look now at other, plausible rules:

($small * x \Rightarrow small$) $\Gamma_1 \in \mathcal{I}(\Sigma_1)$, $\Gamma_2 \subseteq \Sigma_2 \Rightarrow \Gamma_1 \times \Gamma_2 \in \mathcal{I}(\Sigma_1 \times \Sigma_2)$

($big * big \Rightarrow big$) $\Gamma_1 \in \mathcal{F}(\Sigma_1)$, $\Gamma_2 \in \mathcal{F}(\Sigma_2) \Rightarrow \Gamma_1 \times \Gamma_2 \in \mathcal{F}(\Sigma_1 \times \Sigma_2)$

($big * medium \Rightarrow medium$) $\Gamma_1 \in \mathcal{F}(\Sigma_1)$, $\Gamma_2 \in \mathcal{M}^+(\Sigma_2) \Rightarrow \Gamma_1 \times \Gamma_2 \in \mathcal{M}^+(\Sigma_1 \times \Sigma_2)$

($medium * medium \Rightarrow medium$) $\Gamma_1 \in \mathcal{M}^+(\Sigma_1)$, $\Gamma_2 \in \mathcal{M}^+(\Sigma_2) \Rightarrow \Gamma_1 \times \Gamma_2 \in \mathcal{M}^+(\Sigma_1 \times \Sigma_2)$

When we accept all above rules, we can invert ($big * big \Rightarrow big$), as a big product must be composed of big components. Likewise, at least one component of a small product has to be small - see Fact 4.1 (page 14).

We see that these properties give a lot of modularity. We can calculate the consequences of α and α' separately - provided α, α' use disjoint alphabets - and put the results together afterwards. Such properties are particularly interesting for classification purposes, where subclasses are defined with disjoint alphabets.

4.2 Size multiplication and corresponding preferential relations

We turn to those conditions which provide the key to non-monotonic interpolation theorems - see Section 4.3 (page 16). We quote from [GS09c] the following pairwise equivalent conditions $(S * 1)$, $(\mu * 1)$, $(S * 2)$, $(\mu * 2)$, and add a new condition, $(s * s)$, for a principal filter generated by μ :

Definition 4.1

$(S * 1)$ $\Delta \subseteq \Sigma' \times \Sigma''$ is big iff there is $\Gamma = \Gamma' \times \Gamma'' \subseteq \Delta$ s.t. $\Gamma' \subseteq \Sigma'$ and $\Gamma'' \subseteq \Sigma''$ are big

$(\mu * 1)$ $\mu(\Sigma' \times \Sigma'') = \mu(\Sigma') \times \mu(\Sigma'')$

$(S * 2)$ $\Gamma \subseteq \Sigma$ is big $\Rightarrow \Gamma \restriction X' \subseteq \Sigma \restriction X'$ is big - where Σ is not necessarily a product.

$(\mu * 2)$ $\mu(\Sigma) \subseteq \Gamma \Rightarrow \mu(\Sigma \restriction X') \subseteq \Gamma \restriction X'$

$(s * s)$ Let $\Gamma_i \subseteq \Sigma_i$, then $\Gamma_1 \times \Gamma_2 \subseteq \Sigma_1 \times \Sigma_2$ is small iff $\Gamma_1 \subseteq \Sigma_1$ is small or $\Gamma_1 \subseteq \Sigma_1$ is small.

$(\mu * 1)$ and $(s * s)$ are equivalent in the following sense:

Fact 4.1

Let the notion of size satisfy (Opt), (iM), and $(< \omega * s)$, see the tables "Rules on size" in Section 2 (page 4). Then $(\mu * 1)$ and $(s * s)$ are equivalent.

Proof

"$\Rightarrow$":

(1) Let $\Gamma' \subseteq \Sigma'$ be small, we show that $\Gamma' \times \Gamma'' \subseteq \Sigma' \times \Sigma''$ is small. So $\Sigma' - \Gamma' \subseteq \Sigma'$ is big, so by (Opt) and $(\mu * 1)$ $(\Sigma' - \Gamma') \times \Sigma'' \subseteq \Sigma' \times \Sigma''$ is big, so $\Gamma' \times \Sigma'' = (\Sigma' \times \Sigma'') - ((\Sigma' - \Gamma') \times \Sigma'') \subseteq \Sigma' \times \Sigma''$ is small, so by (iM) $\Gamma' \times \Gamma'' \subseteq \Sigma' \times \Sigma''$ is small.

(2) Suppose $\Gamma' \subseteq \Sigma'$ and $\Gamma'' \subseteq \Sigma''$ are not small, we show that $\Gamma' \times \Gamma'' \subseteq \Sigma' \times \Sigma''$ is not small. So $\Sigma' - \Gamma' \subseteq \Sigma'$ and $\Sigma'' - \Gamma'' \subseteq \Sigma''$ are not big. We show that $Z := ((\Sigma' \times \Sigma'') - (\Gamma' \times \Gamma'')) \subseteq \Sigma' \times \Sigma''$ is not big. $Z = (\Sigma' \times (\Sigma'' - \Gamma'')) \cup ((\Sigma' - \Gamma') \times \Sigma'')$.

Suppose $X' \times X'' \subseteq Z$, then $X' \subseteq \Sigma' - \Gamma'$ or $X'' \subseteq \Sigma'' - \Gamma''$. Proof: Let $X' \not\subseteq \Sigma' - \Gamma'$ and $X'' \not\subseteq \Sigma'' - \Gamma''$, but $X' \times X'' \subseteq Z$. Let $\sigma' \in X' - (\Sigma' - \Gamma')$, $\sigma'' \in X'' - (\Sigma'' - \Gamma'')$, consider $\sigma'\sigma''$. $\sigma'\sigma'' \notin (\Sigma' - \Gamma') \times \Sigma''$, as $\sigma' \notin \Sigma' - \Gamma'$, $\sigma'\sigma'' \notin \Sigma' \times (\Sigma'' \times \Gamma'')$, as $\sigma'' \notin \Sigma'' - \Gamma''$, so $\sigma'\sigma'' \notin Z$.

By prerequisite, $\Sigma' - \Gamma' \subseteq \Sigma'$ is not big, $\Sigma'' - \Gamma'' \subseteq \Sigma''$ is not big, so by (iM) no X' with $X' \subseteq \Sigma' - \Gamma'$ is big, no X'' with $X'' \subseteq \Sigma'' - \Gamma''$ is big, so by $(\mu * 1)$ or $(S * 1)$ $Z \subseteq \Sigma' \times \Sigma''$ is not big, so $\Gamma' \times \Gamma'' \subseteq \Sigma' \times \Sigma''$ is not small.

"$\Leftarrow$":

(1) Suppose $\Gamma' \subseteq \Sigma'$ is big, $\Gamma'' \subseteq \Sigma''$ is big, we have to show $\Gamma' \times \Gamma'' \subseteq \Sigma' \times \Sigma''$ is big. $\Sigma' - \Gamma' \subseteq \Sigma'$ is small, $\Sigma'' - \Gamma'' \subseteq \Sigma''$ is small, so by $(s * s)$ $(\Sigma' - \Gamma') \times \Sigma'' \subseteq \Sigma' \times \Sigma''$ is small and $\Sigma' \times (\Sigma'' - \Gamma'') \subseteq \Sigma' \times \Sigma''$ is small, so by $(< \omega * s)$ $(\Sigma' \times \Sigma'') - (\Gamma' \times \Gamma'') = ((\Sigma' - \Gamma') \times \Sigma'') \cup (\Sigma' \times (\Sigma'' - \Gamma'')) \subseteq \Sigma' \times \Sigma''$ is small, so $\Gamma' \times \Gamma'' \subseteq \Sigma' \times \Sigma''$ is big.

(2) Suppose $\Gamma' \times \Gamma'' \subseteq \Sigma' \times \Sigma''$ is big, we have to show $\Gamma' \subseteq \Sigma'$ is big, and $\Gamma'' \subseteq \Sigma''$ is big. By prerequisite, $(\Sigma' \times \Sigma'') - (\Gamma' \times \Gamma'') = ((\Sigma' - \Gamma') \times \Sigma'') \cup (\Sigma' \times (\Sigma'' - \Gamma'')) \subseteq \Sigma' \times \Sigma''$ is small, so by (iM) $\Sigma' \times (\Sigma'' - \Gamma'') \subseteq \Sigma' \times \Sigma''$ is small, so by (Opt) and $(s*s)$ $\Sigma'' - \Gamma'' \subseteq \Sigma''$ is small, so $\Gamma'' \subseteq \Sigma''$ is big, and likewise $\Gamma' \subseteq \Sigma'$ is big.

□

Definition 4.2

Call a relation $\prec$ a GH (= general Hamming) relation iff the following two conditions hold:

$(GH1)$ $\sigma \preceq \tau \wedge \sigma' \preceq \tau' \wedge (\sigma \prec \tau \vee \sigma' \prec \tau') \Rightarrow \sigma\sigma' \prec \tau\tau'$

(where $\sigma \preceq \tau$ iff $\sigma \prec \tau$ or $\sigma = \tau$)

$(GH2)$ $\sigma\sigma' \prec \tau\tau' \Rightarrow \sigma \prec \tau \vee \sigma' \prec \tau'$

$(GH2)$ means that some compensation is possible, e.g., $\tau \prec \sigma$ might be the case, but $\sigma' \prec \tau'$ wins in the end, so $\sigma\sigma' \prec \tau\tau'$.

We use (GH) for $(GH1) + (GH2)$.

Example 4.1

The following are examples of GH relations:

Define on all components X_i a relation $\prec_i$.

(1) The set variant Hamming relation:

Let the relation $\prec$ be defined on $\Pi\{X_i : i \in I\}$ by $\sigma \prec \tau$ iff for all j $\sigma_j \preceq_j \tau_j$, and there is at least one i s.t. $\sigma_i \prec_i \tau_i$.

(2) The counting variant Hamming relation:

Let the relation $\prec$ be defined on $\Pi\{X_i : i \in I\}$ by $\sigma \prec \tau$ iff the number of i such that $\sigma_i \prec_i \tau_i$ is bigger than the number of i such that $\tau_i \prec_i \sigma_i$.

(3) The weighed counting Hamming relation:

Like the counting relation, but we give different (numerical) importance to different i. E.g., $\sigma_1 \prec \tau_1$ may count 1, $\sigma_2 \prec \tau_2$ may count 2, etc.

□

Proposition 4.2

Let $\sigma \prec \tau \Leftrightarrow \tau \notin \mu(\{\sigma, \tau\})$ and $\prec$ be smooth. Then μ satisfies $(\mu * 1)$ (or, by Fact 4.1 (page 14) equivalently $(s*s)$) iff $\prec$ is a GH relation.

Proof

(1) $(\mu * 1)$ entails the GH relation conditions

$(GH1)$: Suppose $\sigma \prec \tau$ and $\sigma' \preceq \tau'$. Then $\tau \notin \mu(\{\sigma, \tau\}) = \{\sigma\}$, and $\mu(\{\sigma', \tau'\}) = \{\sigma'\}$ (either $\sigma' \prec \tau'$ or $\sigma' = \tau'$, so in both cases $\mu(\{\sigma', \tau'\}) = \{\sigma'\}$). As $\tau \notin \mu(\{\sigma, \tau\})$, $\tau\tau' \notin \mu(\{\sigma, \tau\} \times \{\sigma', \tau'\}) =_{(\mu*1)} \mu(\{\sigma, \tau\}) \times \mu(\{\sigma', \tau'\}) = \{\sigma\} \times \{\sigma'\} = \{\sigma\sigma'\}$, so by smoothness $\sigma\sigma' \prec \tau\tau'$.

$(GH2)$: Let $X := \{\sigma, \tau\}$, $Y := \{\sigma', \tau'\}$, so $X \times Y = \{\sigma\sigma', \sigma\tau', \tau\sigma', \tau\tau'\}$. Suppose $\sigma\sigma' \prec \tau\tau'$, so $\tau\tau' \notin \mu(X \times Y) =_{(\mu*1)} \mu(X) \times \mu(Y)$. If $\sigma \not\prec \tau$, then $\tau \in \mu(X)$, likewise if $\sigma' \not\prec \tau'$, then $\tau' \in \mu(Y)$, so $\tau\tau' \in \mu(X \times Y)$, contradiction.

(2) The GH relation conditions generate $(\mu * 1)$.

$\mu(X \times Y) \subseteq \mu(X) \times \mu(Y)$: Let $\tau \in X$, $\tau' \in Y$, $\tau\tau' \notin \mu(X) \times \mu(Y)$, then $\tau \notin \mu(X)$ or $\tau' \notin \mu(Y)$. Suppose $\tau \notin \mu(X)$, let $\sigma \in X$, $\sigma \prec \tau$, so by condition $(GH1)$ $\sigma\tau' \prec \tau\tau'$, so $\tau\tau' \notin \mu(X \times Y)$.

$\mu(X) \times \mu(Y) \subseteq \mu(X \times Y)$: Let $\tau \in X$, $\tau' \in Y$, $\tau\tau' \notin \mu(X \times Y)$, so there is $\sigma\sigma' \prec \tau\tau'$, $\sigma \in X$, $\sigma' \in Y$, so by $(GH2)$ either $\sigma \prec \tau$ or $\sigma' \prec \tau'$, so $\tau \notin \mu(X)$ or $\tau' \notin \mu(Y)$, so $\tau\tau' \notin \mu(X) \times \mu(Y)$.

□

Fact 4.3

(1) Let $\Gamma \subseteq \Sigma$, $\Gamma' \subseteq \Sigma'$, $\Gamma \times \Gamma' \subseteq \Sigma \times \Sigma'$ be small, let $(GH2)$ hold, then $\Gamma \subseteq \Sigma$ is small or $\Gamma' \subseteq \Sigma'$ is small.

(2) Let $\Gamma \subseteq \Sigma$ be small, $\Gamma' \subseteq \Sigma'$, let $(GH1)$ hold, then $\Gamma \times \Gamma' \subseteq \Sigma \times \Sigma'$ is small.

Proof

(1) Suppose $\Gamma \subseteq \Sigma$ is not small, so there is $\gamma \in \Gamma$ and no $\sigma \in \Sigma$ with $\sigma \prec \gamma$. Fix this γ. Consider $\{\gamma\} \times \Gamma'$. As $\Gamma \times \Gamma' \subseteq \Sigma \times \Sigma'$ is small, there is for each $\gamma\gamma'$, $\gamma' \in \Gamma'$ some $\sigma\sigma' \in \Sigma \times \Sigma'$, $\sigma\sigma' \prec \gamma\gamma'$. By $(GH2)$ $\sigma \prec \gamma$ or $\sigma' \prec \gamma'$, but $\sigma \prec \gamma$ was excluded, so for all $\gamma' \in \Gamma'$ there is $\sigma' \in \Sigma'$ with $\sigma' \prec \gamma'$, so $\Gamma' \subseteq \Sigma'$ is small.

(2) Let $\gamma \in \Gamma$, so there is $\sigma \in \Sigma$ and $\sigma \prec \gamma$. By $(GH1)$, for any $\gamma' \in \Gamma'$ $\sigma\gamma' \prec \gamma\gamma'$, so no $\gamma\gamma' \in \Gamma \times \Gamma'$ is minimal.

□

To complete our picture, we repeat from [GS09c] the following very (perhaps too much so - see the discussion there) strong definition and two results (the reader is referred there for proofs):

Definition 4.3

$(GH+)$ $\sigma \preceq \tau \wedge \sigma' \preceq \tau' \wedge (\sigma \prec \tau \vee \sigma' \prec \tau') \Leftrightarrow \sigma\sigma' \prec \tau\tau'$.

(Of course, $(GH+)$ entails (GH).)

Fact 4.4

$(\mu * 1)$ and $(\mu * 2)$ and the usual axioms for smooth relations characterize relations satisfying $(GH+)$.

Proposition 4.5

Interpolation of the form $\phi \vdash \alpha \mid\sim \psi$ exists, if $(\mu * 1)$ and $(\mu * 2)$ hold.

Note

Note that already $(\mu * 1)$ results in a strong independence result in the second scenario: Let $\sigma\rho' \prec \tau\rho'$, then $\sigma\rho'' \prec \tau\rho''$ for all ρ''. Thus, whether $\{\rho''\}$ is small, or medium size (i.e. $\rho'' \in \mu(\Sigma')$), the behaviour of $\Sigma \times \{\rho''\}$ is the same. This we do not have in the first scenario, as small sets may behave very differently from medium size sets. (But, still, their internal structure is the same, only the minimal elements change.) When $(\mu * 2)$ holds, then if $\sigma\sigma' \prec \tau\tau'$ and $\sigma \neq \tau$, then $\sigma \prec \tau$, i.e. we need not have $\sigma' = \tau'$.

4.3 Semantical interpolation

4.3.1 Monotonic interpolation

Definition 4.4

Table 5: Notation and Definitions

Notation and definitions		
	2-valued $\{0,1\}$	many-valued $(V,\leq)$
language $L' \subseteq L$ semantic equivalence of ϕ, ψ	propositional variables $s,\ldots$ $f_\phi = f_\psi$ (or for all m $f_{m,\phi} = f_{m,\psi}$)	
definability of f	$\exists \phi : f_\phi = f$ (or for all m $f_{m,\phi} = f_m$)	
$\Gamma \upharpoonright L'$	(for $\Gamma \subseteq M$) $\Gamma \upharpoonright L' := \{m \upharpoonright L' : m \in \Gamma\}$	
model m M set of all $L-$models	$m : L \to \{0,1\}$	$m : L \to V$
$m \upharpoonright L'$	like m, but restricted to L'	
$m \sim_{L'} m'$	$m \sim_{L'} m'$ iff $\forall s \in L'.m(s) = m'(s)$	
model set of formula ϕ	$M(\phi) \subseteq M$, $f_\phi : M \to \{0,1\}$	$f_\phi : M \to V$
general model set	$M \subseteq M$, $f : M \to \{0,1\}$	$f : M \to V$
f insensitive to L'	$\forall m, m' \in M.(m \sim_{L-L'} m' \Rightarrow f(m) = f(m'))$	
$f^+(m \upharpoonright L')$, $f^-(m \upharpoonright L')$	$f^+(m \upharpoonright L') = max\{f(m') : m' \in M,\ m \sim_{L'} m'\}$ $f^-(m \upharpoonright L') = min\{f(m') : m' \in M,\ m \sim_{L'} m'\}$	
$f \leq g$	$\forall m \in M.f(m) \leq g(m)$	

Let M be the set of models for some language $\mathcal{L}$ with set L of propositional variables. Let $(V,\leq)$ be a finite, totally ordered set (of values). Let $\Gamma \subseteq M$. m, n etc. will be elements of M. As usual, $\upharpoonright$ will denote the restriction of a function to part of its domain, and, by abuse of language, the restrictions of a set of functions.

(1) Let $J \subseteq L$, $f : \Gamma \to V$. Define $f^+(m \upharpoonright J) := max\{f(m') : m \upharpoonright J = m' \upharpoonright J\}$ and $f^-(m \upharpoonright J) := min\{f(m') : m \upharpoonright J = m' \upharpoonright J\}$. (Similarly, if m is defined only on J, the condition is $m' \upharpoonright J = m$, instead of $m \upharpoonright J = m' \upharpoonright J$.)

(2) Call Γ rich iff for all $m, m' \in \Gamma$, $J \subseteq L$ $(m \upharpoonright J) \cup (m' \upharpoonright (L-J)) \in \Gamma$. (I.e., we may cut and paste models.)

(3) Call $f : \Gamma \to V$ insensitive to $J \subseteq L$ iff for all m, n $m \upharpoonright (L-J) = n \upharpoonright (L-J)$ implies $f(m) = f(n)$ - i.e., the values of m on J have no importance for f.

Let $L = J \cup J' \cup J''$ be a disjoint union. If $f : M \to V$ is insensitive to $J \cup J''$, we can define for $m_{J'} : J' \to V$ $f(m_{J'})$ as any $f(m')$ such that $m' \upharpoonright J' = m_{J'}$.

Fact 4.6

Let Γ be rich, $f, g : \Gamma \to V$, $f(m) \leq g(m)$ for all $m \in \Gamma$. Let $L = J \cup J' \cup J''$, let f be insensitive to J, g be insensitive to J''.

Then $f^+(m_{J'}) \leq g^-(m_{J'})$ for all $m_{J'} \in \Gamma \upharpoonright J'$, and any $h : \Gamma \upharpoonright J' \to V$ which is insensitive to $J \cup J''$ is an interpolant iff

$f^+(m_{J'}) \leq h(m_{J'}) \leq g^-(m_{J'})$ for all $m_{J'} \in \Gamma \upharpoonright J'$.

h can then be extended to the full Γ in a unique way, as it is insensitive to $J \cup J''$.

Proof

Let $L = J \cup J' \cup J''$ be a pairwise disjoint union. Let f be insensitive to J, g be insensitive to J''.

$h : \Gamma \to V$ will have to be insensitive to $J \cup J''$, so we will have to define h on $\Gamma \upharpoonright J'$, the extension to Γ is then trivial.

Fix arbitrary $m_{J'} : J' \to V$, $m_{J'} = m \upharpoonright J'$ for some $m \in \Gamma$. We have $f^+(m_{J'}) \leq g^-(m_{J'})$.

Proof: Choose $m_{J''}$ such that $f^+(m_{J'}) = f(m_J m_{J'} m_{J''})$ for any m_J. (Recall that f is insensitive to J.) Let $n_{J''}$ be one such $m_{J''}$. Likewise, choose m_J such that $g^-(m_{J'}) = g(m_J m_{J'} m_{J''})$ for any $m_{J''}$. Let n_J be one such m_J. Consider $n_J m_{J'} n_{J''} \in \Gamma$ (recall that Γ is rich). By definition, $f^+(m_{J'}) = f(n_J m_{J'} n_{J''})$ and $g^-(m_{J'}) = g(n_J m_{J'} n_{J''})$, but by prerequisite $f(n_J m_{J'} n_{J''}) \leq g(n_J m_{J'} n_{J''})$, so $f^+(m_{J'}) \leq g^-(m_{J'})$.

Thus, any h such that h is insensitive to $J \cup J''$ and

(Int) $f^+(m_{J'}) \leq h(m_{J'}) \leq g^-(m_{J'})$

is an interpolant for f and g.

But (Int) is also a necessary condition.

Proof:

Suppose h is insensitive to $J \cup J''$ and $h(m_{J'}) < f^+(m_{J'})$. Let $n_{J''}$ be as above, i.e., $f(m_J m_{J'} n_{J''}) = f^+(m_{J'})$ for any m_J. Then $h(m_J m_{J'} n_{J''}) = h(m_{J'}) < f^+(m_{J'}) = f(m_J m_{J'} n_{J''})$, so h is not an interpolant.

The proof that $h(m_{J'})$ has to be $\leq g^-(m_{J'})$ is analogous.

We summarize:

f and g have an interpolant h, and h is an interpolant for f and g iff h is insensitive to $J \cup J''$ and for any $m_{J'} \in \Gamma \upharpoonright J'$ $f^+(m_{J'}) \leq h(m_{J'}) \leq g^-(m_{J'})$.

□

4.3.2 Non-monotonic interpolation

Proposition 4.7

$(\mu * 1)$ entails semantical interpolation of the form $\phi \mid\!\sim \alpha \mid\!\sim \psi$ in 2-valued non-monotonic logic generated by minimal model sets. (As the model sets might not be definable, syntactic interpolation does not follow automatically.)

Proof

Let the product be defined on $J \cup J' \cup J''$ (i.e., $J \cup J' \cup J''$ is the set of propositional variables in the intended application). Let ϕ be defined on $J' \cup J''$, ψ on $J \cup J'$. See Diagram 4.2 (page 19).

We abuse notation and write $\phi \mid\!\sim \Sigma$ if $\mu(\phi) \subseteq \Sigma$. As usual, $\mu(\phi)$ abbreviates $\mu(M(\phi))$.

For clarity, even if it clutters up notation, we will be precise about where μ is formed. Thus, we write $\mu_{J \cup J' \cup J''}(X)$ when we take the minimal elements in the full product, $\mu_J(X)$ when we consider only the product on J, etc.

Let $\phi \mid\!\sim \psi$, i.e., $\mu_{J \cup J' \cup J''}(\phi) \subseteq M(\psi)$. We show that $X_J \times (\mu_{J \cup J' \cup J''}(\phi) \upharpoonright J') \times X_{J''}$, i.e., that $\mu_{J \cup J' \cup J''}(\phi) \subseteq X_J \times (\mu_{J \cup J' \cup J''}(\phi) \upharpoonright J') \times X_{J''}$, and that $\mu_{J \cup J' \cup J''}(X_J \times (\mu_{J \cup J' \cup J''}(\phi) \upharpoonright J') \times X_{J''}) \subseteq M(\psi)$.

The first property is trivial, we turn to the second. (1) As $M(\phi) = X_J \times M(\phi) \upharpoonright (J' \cup J'')$, $\mu_{J \cup J' \cup J''}(\phi) = \mu_J(X_J) \times \mu_{J' \cup J''}(M(\phi) \upharpoonright (J' \cup J''))$ by $(\mu * 1)$.

(2) By $(\mu * 1)$, $\mu_{J \cup J' \cup J''}(X_J \times (\mu_{J \cup J' \cup J''}(\phi) \upharpoonright J') \times X_{J''}) = \mu_J(X_J) \times \mu_{J'}(\mu_{J \cup J' \cup J''}(\phi) \upharpoonright J') \times \mu_{J''}(X_{J''})$.

So it suffices to show $\mu_J(X_J) \times \mu_{J'}(\mu_{J \cup J' \cup J''}(\phi) \upharpoonright J') \times \mu_{J''}(X_{J''}) \models \psi$.

Proof: Let $\sigma = \sigma_J \sigma_{J'} \sigma_{J''} \in \mu_J(X_J) \times \mu_{J'}(\mu_{J \cup J' \cup J''}(\phi) \upharpoonright J') \times \mu_{J''}(X_{J''})$, so $\sigma_J \in \mu_J(X_J)$.

By $\mu_{J'}(\mu_{J \cup J' \cup J''}(\phi) \upharpoonright J') \subseteq \mu_{J \cup J' \cup J''}(\phi) \upharpoonright J'$, there is $\sigma' = \sigma'_J \sigma'_{J'} \sigma'_{J''} \in \mu_{J \cup J' \cup J''}(\phi)$ s.t. $\sigma'_{J'} = \sigma_{J'}$, i.e. $\sigma' = \sigma'_J \sigma_{J'} \sigma'_{J''}$. As $\sigma' \in \mu_{J \cup J' \cup J''}(\phi)$, $\sigma' \models \psi$.

By (1) and $\sigma_J \in \mu_J(X_J)$ also $\sigma_J \sigma_{J'} \sigma'_{J''} \in \mu_{J \cup J' \cup J''}(\phi)$, so also $\sigma_J \sigma_{J'} \sigma'_{J''} \models \psi$.

But ψ does not depend on J'', so also $\sigma = \sigma_J \sigma_{J'} \sigma_{J''} \models \psi$.

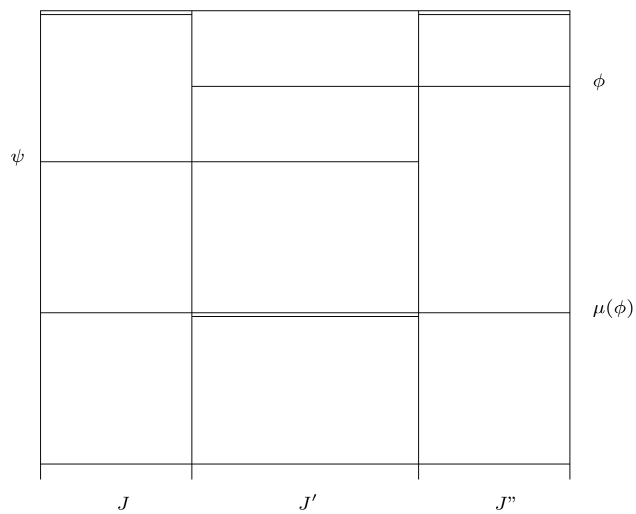
□

Remark 4.8

We can try to extend our result to many-vanued logics. But the we have first to make precise what we want

to do. One approach might be: Let max be the maximal truth value. We look at the set of models where a formula has truth value max, and then look at the minimal models of this set, under some relation. But we can also consider other ideas: we can look at all truth values separately, do minimization for all values separately, etc.

Diagram 4.2



Non-monotonic interpolation
Double lines: interpolant

Remarks for the converse: from interpolation to $(\mu * 1)$

Example 4.2

We show here in (1) and (2) that half of the condition $(\mu * 1)$ is not sufficient for interpolation, and in (3) that interpolation may hold, even if $(\mu * 1)$ fails. When looking closer, the latter is not surprising: μ of sub-products may be defined in a funny way, which has nothing to do with the way μ on the big product is defined.

Consider the language based on p, q, r.

For (1) and (2) define the order $\prec$ on sequences of length 3 by $\neg p \neg q \neg r \prec p \neg q \neg r$, leave all other 3-sequences incomparabel.

Let $\phi = \neg q \wedge \neg r$, $\psi = \neg p \wedge \neg q$, so $\mu(\phi) = \neg p \wedge \neg q \wedge \neg r$, and $\phi \mid\sim \psi$. Suppose there is α, $\phi \mid\sim \alpha \mid\sim \psi$, α written with q only, so α is equivalent to FALSE, TRUE, q, or $\neg q$. $\phi \not\mid\sim FALSE$, $\phi \not\mid\sim q$. $TRUE \not\mid\sim \psi$, $\neg q \not\mid\sim \psi$. Thus, there is no such α, and $\mid\sim$ has no interpolation. We show in (1) and (2) that we can make both directions of $(\mu * 1)$ true separately, so they do not suffice to obtain interpolation.

(1) We make $\mu(X \times Y) \subseteq \mu(X) \times \mu(Y)$ true, but not the converse.

Do not order any sequences of length 2 or 1, i.e. μ is there always identity. Thus, $\mu(X \times Y) \subseteq X \times Y = \mu(X) \times \mu(Y)$ holds trivially.

For (2) and (3), consider the following ordering $<$ between sequences: $\sigma < \tau$ iff there is $\neg x$ is σ, x in τ, but for no y y in σ, $\neg y$ in τ. E.g., $\neg p < p$, $\neg pq < pq$, $\neg p \neg q < pq$, but $\neg pq \not< p \neg q$.

(2) We make $\mu(X \times Y) \supseteq \mu(X) \times \mu(Y)$ true, but not the converse.

We order all sequences of length 1 or 2 by $<$.

Suppose $\sigma \in X \times Y - \mu(X \times Y)$. Case 1: $X \times Y$ consists of sequences of length 2. Then, by definition, $\sigma \notin \mu(X) \times \mu(Y)$. Case 2: $X \times Y$ consists of sequences of length 3. Then $\sigma = p\neg q\neg r$, and there is $\tau = \neg p\neg q\neg r \in X \times Y$. So $\{p, \neg p\} \subseteq X$ or $\{p\neg q, \neg p\neg q\} \subseteq X$, but in both cases $\sigma \restriction X \notin \mu(X)$.

Finally, note that $\mu(TRUE) \not\subseteq \{\neg p\neg q\neg r\}$, so full $(\mu * 1)$ does not hold.

(3) We make interpolation hold, but $\mu(X) \times \mu(Y) \not\subseteq \mu(X \times Y)$:

We order all sequences of length 3 by $<$. Shorter sequences are made incomparabel, so for shorter sequences $\mu(X) = X$.

Obviously, in general $\mu(X) \times \mu(Y) \not\subseteq \mu(X \times Y)$.

But the proof of Proposition 4.7 (page 18) goes through as above, only directly, without the use of factorizing and taking μ of the factors.

□

4.4 Language change

Independence of language fragments gives us the following perspectives:

(1) it makes independent and parallel treatment of fragments possible, and offers thus efficient treatment in applications (descriptive logics etc.).

(2) it results in new rules similar to the classical ones like AND, OR, Cumulativity, etc. We can thus obtain postulates about reasonable behaviour, but also classification by those rules, see Table 6 (page 23), Scenario 2, Logical property.

(3) it sheds light on notions like “ceteris paribus”, which we saw in the context of obligations, see [GS08g].

(4) it clarifies notions like “normal with respect to ϕ, but not ψ”

(5) it helps to understand e.g. inheritance diagrams where arrows make other information accessible, and we need an underlying mechanism to combine bits of information, given in different languages.

4.5 A relevance problem

Consider the formula $\phi := a \wedge \neg a \wedge b$. Then $M(\phi) = \emptyset$. But we cannot recover where the problem came from, and this results in the EFQ rule. We now discuss one, purely algebraic, approach to remedy.

Consider 3 valued models, with a new value b for both, in addition to t and f. Above formula would then have the model $m(a) = b$, $m(b) = t$. So there is a model, EFQ fails, and we can recover the culprit.

To have the usual behaviour of $\wedge$ as intersection, it might be good to change the definition so that $m(x) = b$ is always a model. Then $M(b) = \{m(b) = t, m'(b) = b\}$, $M(\neg b) = \{m(b) = f, m'(b) = b\}$, and $M(b \wedge \neg b) = \{m'(b) = b\}$.

It is not yet clear which version to choose, and we have no syntactic characterization.

Other idea:

Use meaningless models. Take a conjunction of literals. $m(a) = t$ and $m(a) = x$ is a model if there is only a in the conjunction, $m(a) = f$ and $m(a) = x$ if there is only $\neg a$ in the conjunction, $m(a) = *$ if both are present,

and all models, if none is present. Thus there is always a model, and we can isolate the contradictory parts: there, only $m(a) = x$ is present.

4.6 Small subspaces

When considering small subsets in nonmonotonic logic, we neglect small subsets of models. What is the analogue when considering small subspaces, i.e. when $J = J' \cup J''$, with J'' small in J in nonmonotonic logic?

It is perhaps easiest to consider the relation based approach first. So we have an order on $\Pi J'$ and one on $\Pi J''$, J'' is small, and we want to know how to construct a corresponding order on ΠJ. Two solutions come to mind:

- a less radical one: we make a lexicographic ordering, where the one on $\Pi J'$ has precedence over the one on $\Pi J''$,
- a more radical one: we totally forget about the ordering of $\Pi J''$, i.e. we do as if the ordering on $\Pi J''$ were the empty set, i.e. $\sigma'\sigma'' \prec \tau'\tau''$ iff $\sigma' \prec \tau'$ and $\sigma'' = \tau''$.

  We call this condition $forget(J'')$.

The less radical one is already covered by our relation conditions (GH). The more radical one is probably more interesting. Suppose ϕ' is written in language J', ϕ'' in language J'', we then have

$\phi' \wedge \phi'' \mid\!\sim \psi' \wedge \psi''$ iff $\phi' \mid\!\sim \psi'$ and $\phi'' \vdash \psi''$.

This approach, is of course the same as considering on the small coordinate only ALL as a big subset, (see the lines $x * 1/1 * x$ in Table 6 (page 23)), so, in principle, we get nothing new.

5 Revision and distance relations

We will look here into distance based theory revision a la AGM, see [AGM85] and [LMS01], and also [Sch04] for more details. First, we introduce some notation, and give a result taken from [GS09c] (slightly modified).

Definition 5.1

Let d be a distance on some product space $X \times Y$, and its components. (We require of distances only that they are comparable, that $d(x, y) = 0$ iff $x = y$, and that $d(x, y) \geq 0$.)

d is called a generalized Hamming distance (GHD) iff it satisfies the following two properties:

$(GHD1)$ $d(\sigma, \tau) \leq d(\alpha, \beta)$ and $d(\sigma', \tau') \leq d(\alpha', \beta')$ and $(d(\sigma, \tau) < d(\alpha, \beta)$ or $d(\sigma', \tau') < d(\alpha', \beta')) \Rightarrow d(\sigma\sigma', \tau\tau') < d(\alpha\alpha', \beta\beta')$

$(GHD2)$ $d(\sigma\sigma', \tau\tau') < d(\alpha\alpha', \beta\beta') \Rightarrow d(\sigma, \tau) < d(\alpha, \beta)$ or $d(\sigma', \tau') < d(\alpha', \beta')$

Definition 5.2

Given a distance d, define for two sets X, Y

$X \mid Y := \{y \in Y : \exists x \in X(\neg\exists x' \in X, y' \in Y.d(x', y') < d(x, y))\}$.

We assume that $X \mid Y \neq \emptyset$ if $X, Y \neq \emptyset$. Note that this is related to the consistency axiom of AGM theory revision: revising by a consistent formula gives a consistent result. The assumption may be wrong due to infinite descending chains of distances.

Definition 5.3

Given $\mid$ on models, we can define an AGM revision operator $*$ as follows:

$T * \phi := Th(M(T) \mid M(\phi))$

where T is a theory, and $Th(X)$ is the set of formulas which hold in all $x \in X$.

It was shown in [LMS01] that a revision operator thus defined satisfies the AGM revision postulates.

We have a result analogous to the relation case:

Fact 5.1

Let $\mid$ be defined by a generalized Hamming distance, then $\mid$ satisfies

$(\mid *)$ $(\Sigma_1 \times \Sigma_1') \mid (\Sigma_2 \times \Sigma_2') = (\Sigma_1 \mid \Sigma_2) \times (\Sigma_1' \mid \Sigma_2')$.

Proof

"$\subseteq$":

Suppose $d(\sigma\sigma', \tau\tau')$ is minimal. If there is $\alpha \in \Sigma_1$, $\beta \in \Sigma_2$ s.t. $d(\alpha, \beta) < d(\sigma, \tau)$, then $d(\alpha\sigma', \beta\tau') < d(\sigma\sigma', \tau\tau')$ by $(GHD1)$, so $d(\sigma, \tau)$ and $d(\sigma', \tau')$ have to be minimal.

"$\supseteq$":

For the converse, suppose $d(\sigma, \tau)$ and $d(\sigma', \tau')$ are minimal, but $d(\sigma\sigma', \tau\tau')$ is not, so $d(\alpha\alpha', \beta\beta') < d(\sigma\sigma', \tau\tau')$ for some $\alpha\alpha', \beta\beta'$, then $d(\alpha, \beta) < d(\sigma, \tau)$ or $d(\alpha', \beta') < d(\sigma', \tau')$ by $(GHD2)$, contradiction.

□

These properties translate to logic as follows:

Corollary 5.2

If ϕ and ψ are defined on a separate language from that of ϕ' and ψ', and the distance satisfies $(GHD1)$ and $(GHD2)$, then for revision holds:

$(\phi \wedge \phi') * (\psi \wedge \psi') = (\phi * \psi) \wedge (\phi' * \psi')$.

5.1 Interpolation for distance based revision

The limiting condition (consistency) imposes a strong restriction: Even for $\phi * TRUE$, the result may need many variables (those in ϕ).

Lemma 5.3

Let $\mid$ satisfy $(\mid *)$.

Let $J \subseteq L$, ρ be written in sublanguage J, let ϕ, ψ be written in $L - J$, let ϕ', ψ' be written in $J' \subseteq J$.

Let $(\phi \wedge \phi') * (\psi \wedge \psi') \vdash \rho$, then $\phi' * \psi' \vdash \rho$.

(This is suitable interpolation, but we also need to factorize the revision construction.)

Proof

$(\phi \wedge \phi') * (\psi \wedge \psi') = (\phi * \psi) \wedge (\phi' * \psi')$ by $(\mid *)$. So $(\phi \wedge \phi') * (\psi \wedge \psi') \vdash \rho$ iff $(\phi * \psi) \wedge (\phi' * \psi') \vdash \rho$, but $(\phi * \psi) \wedge (\phi' * \psi') \vdash \rho$ iff $(\phi' * \psi') \vdash \rho$, as ρ contains no variables of ϕ' or ψ'. □

6 Summary of properties

$pr(b) = b$ means: the projection of a big set on one of its coordinates is big again.

Note that $A \times B \subseteq X \times Y$ big $\Rightarrow$ $A \subseteq X$ big etc. is intuitively better justified than the other direction, as the proportion might increase in the latter, decrease in the former. Cf. the table "Rules on size", Section 2 (page 4), "increasing proportions".

Multiplication laws								
Multiplication law	Scenario 1 (see Diagram 3.1 (page 11))			Scenario 2 ($*$ symmetrical, only 1 side shown) (see Diagram 4.1 (page 12))				
	Corresponding algebraic addition property	Logical property	Relation property	Algebraic property ($\Gamma_i \subseteq \Sigma_i$)	Logical property α, β in $\mathcal{L}_1$, α', β' in $\mathcal{L}_2$ $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ (disjoint)	Interpolation		
						Multiplic. law	Relation property	Inter-polation
Non-monotonic logic								
$x * 1 \Rightarrow x$	trivial			$\Gamma_1 \in \mathcal{F}(\Sigma_1) \Rightarrow$ $\Gamma_1 \times \Sigma_2 \in \mathcal{F}(\Sigma_1 \times \Sigma_2)$	$\alpha \mid\!\sim_{\mathcal{L}_i} \beta \Rightarrow \alpha \mid\!\sim_{\mathcal{L}} \beta$ $\alpha \mid\!\sim_{\mathcal{L}} \beta$			
$1 * x \Rightarrow x$	trivial							
$x * s \Rightarrow s$	(iM) $A \subseteq B \in \mathcal{I}(X) \Rightarrow A \in \mathcal{I}(X)$	$\alpha \mid\!\sim \neg\beta \Rightarrow$ $\alpha \mid\!\sim \neg\beta \vee \gamma$	-	dual to $x * 1 \Rightarrow 1$	$\alpha \mid\!\sim_{\mathcal{L}_1} \beta \Rightarrow \alpha \mid\!\sim_{\mathcal{L}} \beta$			
$s * x \Rightarrow s$	$(eM\mathcal{I})$ $X \subseteq Y \Rightarrow \mathcal{I}(X) \subseteq \mathcal{I}(Y)$, $X \subseteq Y \Rightarrow$ $\mathcal{F}(Y) \cap \mathcal{P}(X) \subseteq \mathcal{F}(X)$	$\alpha \wedge \beta \mid\!\sim \neg\gamma \Rightarrow$ $\alpha \mid\!\sim \neg\beta \vee \neg\gamma$	-					
$b * b \Rightarrow b$	$(< \omega * s)$, $(\mathcal{M}^+_\omega)$ (3) $A \in \mathcal{F}(X), X \in \mathcal{F}(Y) \Rightarrow$ $A \in \mathcal{F}(Y)$	$\alpha \mid\!\sim \beta$, $\alpha \wedge \beta \mid\!\sim \gamma$ $\Rightarrow \alpha \mid\!\sim \gamma$	- (Filter)	$\Gamma_1 \in \mathcal{F}(\Sigma_1), \Gamma_2 \in \mathcal{F}(\Sigma_2) \Rightarrow$ $\Gamma_1 \times \Gamma_2 \in \mathcal{F}(\Sigma_1 \times \Sigma_2)$	$\alpha \mid\!\sim_{\mathcal{L}_1} \beta$, $\alpha' \mid\!\sim_{\mathcal{L}_2} \beta' \Rightarrow$ $\alpha \wedge \alpha' \mid\!\sim_{\mathcal{L}} \beta \wedge \beta'$	$b * b \Leftrightarrow b$: $(\mu * 1)$ $\Leftrightarrow$ $(s * s)$	(GH)	$\mid\!\sim \circ \mid\!\sim$
$b * m \Rightarrow m$	$(< \omega * s)$, $(\mathcal{M}^+_\omega)$ (2) $A \in \mathcal{M}^+(X), X \in \mathcal{F}(Y) \Rightarrow$ $A \in \mathcal{M}^+(Y)$	$\alpha \not\mid\!\sim \neg\beta$, $\alpha \wedge \beta \mid\!\sim \gamma$ $\Rightarrow \alpha \not\mid\!\sim \neg\beta \vee \neg\gamma$	- (Filter)	$\Gamma_1 \in \mathcal{F}(\Sigma_1), \Gamma_2 \in \mathcal{M}^+(\Sigma_2) \Rightarrow$ $\Gamma_1 \times \Gamma_2 \in \mathcal{M}^+(\Sigma_1 \times \Sigma_2)$	$\alpha \not\mid\!\sim_{\mathcal{L}_1} \neg\beta$, $\alpha' \mid\!\sim_{\mathcal{L}_2} \beta' \Rightarrow$ $\alpha \wedge \alpha' \not\mid\!\sim_{\mathcal{L}} \neg\beta \vee \beta'$			
$m * b \Rightarrow m$	$(< \omega * s)$, $(\mathcal{M}^+_\omega)$ (1) $A \in \mathcal{F}(X), X \in \mathcal{M}^+(Y) \Rightarrow$ $A \in \mathcal{M}^+(Y)$	$\alpha \mid\!\sim \beta$, $\alpha \wedge \beta \not\mid\!\sim \neg\gamma$ $\Rightarrow \alpha \not\mid\!\sim \neg\beta \vee \neg\gamma$	- (Filter)					
$m * m \Rightarrow m$	$(\mathcal{M}^{++})$ $A \in \mathcal{M}^+(X), X \in \mathcal{M}^+(Y)$ $\Rightarrow A \in \mathcal{M}^+(Y)$	Rational Monotony	ranked	$\Gamma_1 \in \mathcal{M}^+(\Sigma_1), \Gamma_2 \in \mathcal{M}^+(\Sigma_2)$ $\Rightarrow$ $\Gamma_1 \times \Gamma_2 \in \mathcal{M}^+(\Sigma_1 \times \Sigma_2)$	$\alpha \not\mid\!\sim_{\mathcal{L}_1} \neg\beta$, $\alpha' \mid\!\sim_{\mathcal{L}_2} \neg\beta' \Rightarrow$ $\alpha \wedge \alpha' \not\mid\!\sim_{\mathcal{L}} \neg\beta \vee \neg\beta'$			
$b * b \Leftrightarrow b +$ $pr(b) = b$					$\alpha \mid\!\sim \beta \Rightarrow \alpha \upharpoonright \mathcal{L}_1 \mid\!\sim \beta \upharpoonright \mathcal{L}_1$ and $\alpha \mid\!\sim_{\mathcal{L}_1} \beta$, $\alpha' \mid\!\sim_{\mathcal{L}_2} \beta' \Rightarrow$ $\alpha \wedge \alpha' \mid\!\sim_{\mathcal{L}} \beta \wedge \beta'$		$(GH+)$	$\vdash \circ \mid\!\sim$
J' small					$\alpha \wedge \alpha' \mid\!\sim \beta \wedge \beta' \Leftrightarrow$ $\alpha \mid\!\sim \beta$, $\alpha' \vdash \beta'$		$forget(J')$	-
Theory revision								
				$(\mid *)$: $(\Sigma_1 \times \Sigma'_1) \mid (\Sigma_2 \times \Sigma'_2) =$ $(\Sigma_1 \mid \Sigma_2) \times (\Sigma'_1 \mid \Sigma'_2)$	$(\phi \wedge \phi') * (\psi \wedge \psi') =$ $(\phi * \psi) \wedge (\phi' * \psi')$		(GHD)	$(\phi \wedge \phi') * (\psi \wedge \psi') \vdash \rho$ $\Rightarrow \phi' * \psi' \vdash \rho$ ϕ, ψ in J, ϕ', ψ', ρ in $L - J$

Table 6: Multiplication laws

mul

References

[AGM85] C.Alchourron, P.Gardenfors, D.Makinson, "On the Logic of Theory Change: partial meet contraction and revision functions", Journal of Symbolic Logic, Vol. 50, pp. 510-530, 1985

[GS08c] D.Gabbay, K.Schlechta, "Roadmap for preferential logics", Journal of applied nonclassical logic, Hermes, Cachan, France, Vol. 19/1, pp. 43-95, 2009, see also hal-00311941, arXiv 0808.3073

[GS08g] D.Gabbay, K.Schlechta, "A semantics for obligations", submitted, hal-00339393, arXiv 0811.2754

[GS09a] D.Gabbay, K.Schlechta, "Size and logic", arXiv 0903.1367, Review of Symbolic Logic, Vol. 2, No. 2, pp. 396-413

[GS09c] D.Gabbay, K.Schlechta, "Semantic interpolation", submitted, preliminary version: arXiv.org 0906.4082

[LMS01] D.Lehmann, M.Magidor, K.Schlechta: "Distance Semantics for Belief Revision", Journal of Symbolic Logic, Vol.66, No. 1, March 2001, *p.* 295-317

[Sch04] K.Schlechta: "Coherent Systems", Elsevier, Amsterdam, 2004

[Sch95-1] K.Schlechta: "Defaults as generalized quantifiers", Journal of Logic and Computation, Oxford, Vol.5, No.4, p.473-494, 1995